On the finiteness of log surfaces

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Abstract

We prove that a log surface has only finitely many weakly log canonical projective models with klt singularities up to log isomorphism, by reducing the problem to the boundedness of their polarization.

Introduction

One of the most powerful tools of modern birational geometry is the Minimal Model Program (MMP):

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X_{\min},$$

which for a given (smooth) projective variety X produces a «good» variety $X_{\min}$ called a *minimal model*, that is, a ($\mathbb{Q}$-factorial) projective variety with *mild singularities* (i.e. terminal singularities) such that the canonical divisor K is *nef* (i.e. $K \cdot C \geq 0$ for all curves $C \subset X_{\min}$). It is essential to raise the question about the number of isomorphism classes of minimal models [Mat02, Conjecture 12.3.6.].

Conjecture 1

The number of projective minimal models in a fixed birational class is finite up to isomorphism.

It is natural to extend this conjecture to a class of crepant birationally equivalent log pairs, namely a *0-class* (cf. Conjecture 2).

0-class

We say that a log pair (X_α, B_α) is *crepant birationally equivalent* to (X, B) if the varieties X_α and X are birationally equivalent, and for each common log resolution (Y, D) of these pairs the following equalities hold:

$$K_Y + D = f^*(K_X + B) = f_\alpha^*(K_{X_\alpha} + B_\alpha),$$
$$B = f_*D, \quad B_\alpha = (f_\alpha)_*D,$$

where $f : Y \to X$ and $f_\alpha : Y \to X_\alpha$ are the corresponding log resolutions. The class of all log pairs crepant birationally equivalent to (X, B) is called the *0-class* of the log pair (X, B).

Bounded polarization

A class $\mathfrak{D}$ of projective varieties X_α has *bounded polarization* if there exists a positive integer $N \in \mathbb{N}$ and very ample Cartier divisors H_α on X_α such that $H_\alpha^{\dim X_\alpha} \leq N$ for all X_α in $\mathfrak{D}$.

Main result

Let (X, B) be a projective Calabi-Yau log surface, i.e. $K_X + B \equiv 0$. Consider the class $\mathfrak{C}$ of all projective wlc klt models (X_α, B_α) that are crepant birationally equivalent to (X, B). Let $\mathfrak{D}$ be the class of all varieties X_α corresponding to the pairs (X_α, B_α) in $\mathfrak{C}$.

Main Theorem

If the class $\mathfrak{D}$ has bounded polarization, then the class $\mathfrak{C}$ contains only finitely many log surfaces up to log isomorphism.

Proof (idea)

By standard arguments, one can construct a good projective family $(\mathcal{X}, \mathcal{B}) \to S$ such that all log pairs in $\mathfrak{C}$ are isomorphic to some fibers of this family. The cornerstone of the proof is the existence of a log surface (Y, D) and an open non-empty subset $U \subseteq S$ such that $(\mathcal{X}, \mathcal{B}) \times_S U \approx (Y, D) \times_{\mathbb{C}} U$. Then the finiteness of log pairs in the family follows from Noetherian induction.

Step 0. It is well-known that a log surface (X, B) has a unique projective wlc trm model in its 0-class up to log isomorphism.

Step 1. We can assume there is a good projective family $(\mathcal{X}/S, \mathcal{B})$ such that all log pairs in $\mathfrak{C}$ are isomorphic to some fibers of the family.

Step 2. Let $\pi : (\mathcal{X}', \mathcal{B}') \to (\mathcal{X}, \mathcal{B})$ be a terminalization, so that $K_{\mathcal{X}'} + \mathcal{B}' = \pi^*(K_{\mathcal{X}} + \mathcal{B})$. After an étale base change, we suppose that $(\mathcal{X}'/S, \mathcal{B}')$ is a trivial family.

Step 3. Let $\mathcal{E} = \mathrm{Exc}(\pi)$. Then we run MMP for $K_{\mathcal{X}'} + \mathcal{B}' + \epsilon\mathcal{E}$ over S, and show that at each step the triviality of the family is preserved.

Example of a 0-class, $B = 0$

Let X be a smooth projective K3 surface containing infinitely many distinct (-2)-curves C_i, that is, smooth rational curves with $C_i^2 = -2$. Let $f_i : X \to X_i$ be the contraction of C_i. Then each log pair $(X_i, 0)$ is a klt Calabi-Yau pair in the 0-class of $(X, 0)$.

Example of a 0-class, $B \neq 0$

Let $X = \mathbb{P}^2_{\mathbb{C}}$ and $B = \frac{1}{2}\sum_{i=1}^{6} L_i$, where $\{L_i\}_{i=1}^6$ is the set of six lines passing through four points in general position. Then (X, B) is a klt Calabi-Yau pair. Let $f_{ij} : X_{ij} \to X$ be the blow-up of a point $p \in L_i \cap L_j$, and let B_{ij} be the $\mathbb{R}$-divisor defined by $K_{X_{ij}} + B_{ij} = f_{ij}^*(K_X + B)$. Then for all $i \neq j$ the log pair (X_{ij}, B_{ij}) is a klt Calabi-Yau pair in the 0-class of (X, B).

Wlc models

A log pair (X, B) is called a *weakly log canonical model (wlc model)* if the following hold:

- X is a proper variety, and B is a boundary.
- (X, B) has log canonical singularities.
- $K_X + B$ is nef.

If (X, B) has klt (resp. terminal) singularities, then (X, B) is called a *wlc klt model* (resp. a *wlc trm model*). If, in addition, $K_X + B \equiv 0$, then (X, B) is called a *Calabi-Yau pair* (0-pair).

Conjecture 2

The number of projective wlc klt models in a fixed 0-class is finite up to log isomorphism.

Applications

Conjecture 2 holds for log surfaces.

Proof (idea)

The most delicate case of Conjecture 2 for log surfaces (X, B) is when $K_X + B \equiv 0$, namely, *Calabi–Yau pairs*. If $B \neq 0$ (resp. $B = 0$) then $\mathfrak{D}$ has bounded polarization due to [Ale94, Theorem 6.9] (resp.[Kaw97, Theorem 2.1]). Then, their finiteness up to isomorphism follows from the main theorem.

References

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